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Iterated quasicomponents of subspaces of rational continua

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Abstract

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A space is *rational* if the collection of all open sets with at most countable boundary is a basis. We construct an example of a rational metrizable continuum X containing a point x such that $\sup\{\mathrm{nc}(Y, x)\colon x \in Y \subseteq X\} = \omega_1$. This answers a question of Lelek.

Keywords: Continuum, index of nonconnectedness.

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Introduction

A space is called *rational* if the collection of all open sets with at most countable boundary is a basis. For a space X and for $x \in X$ define $\mathrm{nc}(X, x)$, the *index of nonconnectedness* X at x, to be the smallest ordinal α such that the component of X at x is obtained by iterating the taking of quasicomponents α many times, see below for a precise definition. It is easy to see that $\mathrm{nc}(X, X) < w(X)^+$, in particular that $\mathrm{nc}(X, x) < \omega_1$ if X is separable metrizable.

Lelek, [1], has proved that if X is a rational metrizable continuum then

$$\sup\{\mathrm{nc}(Y, y)\colon y \in Y\} < \omega_1 \quad \text{for all } Y \subseteq X, \tag{*}$$

and has asked, [2, P6] if this can be improved to

$$\sup\{\mathrm{nc}(Y, y)\colon y \in Y \subseteq X\} < \omega_1. \tag{**}$$

We show that even a slightly weaker version of (**) can fail.

Example. There is a rational metrizable continuum X containing a point x such that

$$\sup\{\mathrm{nc}(Y, x)\colon x \in Y \subseteq X\} = \omega_1. \tag{***}$$

Our continuum X is not locally connected, but otherwise is very nice since it is the union of countably many straight-line segments in the plane with a common endpoint. Also, we prove something formally stronger than (***) by finding for each $\alpha<\omega_1$ a $Y\subseteq X$ with $\mathrm{nc}(Y,x)=\alpha$ whose component at x is $\{x\}$; moreover Y itself is nice since it is both a G_δ and an F_σ in X.

Definitions

The *component* of X at x, denoted by $C(X,x)$ is the biggest connected subspace of X that contains x, and the *quasicomponent* of X at x, denoted by $Q(X,x)$, is the intersection of all clopen (=closed and open) subsets of X that contain x. Of course $C(X,x)\subseteq Q(X,x)$, and $C(X,x)\neq Q(X,x)$ is possible. One can obtain $C(X,x)$ by iterating Q as follows: For $\alpha\in\mathrm{Ord}$ (the class of ordinals) define $Q^\alpha(X,x)$ as follows

$$Q^0(X,x)=X,\qquad Q^{\alpha+1}(X,x)=Q(Q^\alpha(X,x))\quad\text{for }\alpha\in\mathrm{Ord},$$

and

$$Q^\lambda(X,x)=\bigcap_{\alpha<\lambda}Q^\alpha(X,x)\quad\text{if }\lambda\in\mathrm{Ord}\text{ is a limit}$$

or, equivalently

$$Q^0(X,x)=X\quad\text{and}\quad Q^\alpha(X,x)=\bigcap_{\xi<\alpha}Q(Q^\xi(X,x),x)\quad\text{if }\alpha>0.$$

Since $Q^\alpha(X,x)\supseteq Q^\beta(X,x)$ if $\alpha<\beta$, and since each $Q^\alpha(X,x)$ is closed, there must be $\alpha<w(X)^+$ (the smallest cardinal (=initial ordinal) greater than the weight of X) such that $Q^\alpha(X,x)=Q^{\alpha+1}(X,x)$. The smallest such α is denoted by $\mathrm{nc}(X,x)$, the *index of nonconnectedness* of X at x.

The example

Let Π denote the Euclidean plane, equipped with polar coordinates $\langle r;\phi\rangle$. For $x,y\in\Pi$ let xy denote the closed straight-line segment with endpoints x and y. Let $\mathbb{N}$ be the positive integers. Define

$$\mathbf{0}=\langle 0;0\rangle,\qquad b=\langle 1;0\rangle,\qquad b_n=\langle 1;2^{-n}\rangle,\quad\text{and}\quad p_n=\langle 2^{-n};2^{-n}\rangle\quad(n\in\mathbb{N});$$
$$T_n=\{\langle r;\phi\rangle: 0\le r\le 2^{-n}\text{ and }2^{-n}\le\phi\le 2^{-n+1}\}\quad(n\in\mathbb{N}).$$

Note that $p_n\in\mathbf{0}b_n$ and $p_n\in T_n$ for all $n\in\mathbb{N}$. We claim that there is a countable $C\subseteq\Pi$ such that if

$$X=\bigcup_{y\in C}\mathbf{0}y,\quad\text{and}\quad X_n=X\cap T_n\quad\text{for }n\in\mathbb{N},$$

then X is compact and

(I) for all $n\in\mathbb{N}$ there is a homeomorphism from X onto X_n that sends $\mathbf{0}$ to $\mathbf{0}$ and b to p_n;

(II) $X-(X_n-\{0\})$ is compact for all $n\in\mathbb{N}$.

Proof that such C exists. Let

$$T=\{\langle r;\,\phi\rangle\colon 0\leqslant r\leqslant 1 \text{ and } 0\leqslant \phi<1\},$$

and for $n\in\mathbb{N}$ define an embedding $e_n: T\to T_n$ by

$$e_n(\langle r;\,\phi\rangle)=\langle 2^{-n}r;\, 2^{-n}(1+\phi)\rangle,\quad \langle r;\,\phi\rangle\in T.$$

Note that

(1) $e_n(b)=p_n$, and $e_n^{\to}0x=0e_n^{\to}(x)$ for $x\in T$, for $n\in\mathbb{N}$. Define a sequence $\langle C_k\rangle_{n\in\mathbb{N}}$ of countable subsets of Π as follows

$$C_1=\{b\}\cup\{b_n: n\in\mathbb{N}\},\quad\text{and}\quad C_{k+1}=\bigcup_{n\in\mathbb{N}} e_n^{\to}C_k.$$

We will show that $C=\bigcup_{k\in\mathbb{N}}C_k$ is as required. It should be clear that (I) holds because of (1). To prove that X is compact it suffices to show that $C\cup\{0\}$ is compact since X is a continuous image of $(C\cup\{0\})\times[0,1]$: If U_ε denotes the ε-ball $\{\langle r;\,\phi\rangle\in\Pi\colon r<\varepsilon\}$ around $\mathbf{0}$, then clearly $C-U_\varepsilon$ is the union of finitely many copies of the compact set C_1, for each $\varepsilon>0$. This argument also establishes (II).

Proof that X is as required. Clearly X is (arcwise) connected. Also, X is rational since C is countable. We prove that for each $\alpha<\omega_1$ the following holds.

$Q(\alpha)$: there is a subspace Y of X with $\mathbf{0}\in Y$ such that

(1) $C(Y,\mathbf{0})=\{\mathbf{0}\}$;

(2) $\mathrm{nc}(Y,\mathbf{0})=\alpha$;

(3) $Q^{\xi}(Y\cup\{b\},\mathbf{0})=Q^{\xi}(Y,\mathbf{0})\cup\{b\}$ for $\xi\leqslant\alpha$; and

(4) Y is both F_σ and G_δ.

Clearly $Y=\{\mathbf{0}\}$ witnesses $Q(0)$.

Now let $\gamma\in[1,\omega_1)$, and assume $Q(\alpha)$ has been proved for $\alpha<\gamma$. We will treat the two cases γ is a successor and γ is a limit (partially) simultaneously by considering a function $a:\mathbb{N}\to[0,\gamma)$, nondecreasing, with $\gamma=\sup_n(a(n)+1)$. For each $n\in\mathbb{N}$ we have $Q(a(n))$, hence because of (I) we can find $Y_n\subseteq T_n$ such that (1)-(4) hold with Y_n, $a(n)$, p_n instead of Y, α, b.

We claim that

$$Y=\bigcup_{n\in\mathbb{N}}(Y_n\cup p_nb_n)$$

witnesses that $Q(\gamma)$ holds. It should be clear that (4) holds. For each $\alpha\leqslant\gamma$ define

$$Y^{\alpha}=\bigcup\{Q^{\alpha}(Y_n,\mathbf{0})\cup p_nb_n: n\in\omega \text{ and } a(n)\geqslant\alpha\}.$$

We will prove that for all $\alpha\leqslant\gamma$ the following equalities hold

(A) $Q^{\alpha}(Y,\mathbf{0})=Y^{\alpha}$; and

(B) $Q^{\alpha}(Y\cup\{b\},\mathbf{0})=Y^{\alpha}\cup\{b\}$.

Before proving this let us show how it follows that Y is as required: Obviously (A) and (B) imply (3) of $Q(\gamma)$. Also, for each $n\in\mathbb{N}$ we have $a(n)<\gamma$, hence $Q^{\gamma}(Y_n,\mathbf{0})=\{\mathbf{0}\}$, hence $Q^{\gamma}(Y,\mathbf{0})=\{\mathbf{0}\}$ by (A). This proves (1) of $Q(\gamma)$ and also proves that

$\operatorname{nc}(Y, 0) \leq \gamma$. At the other hand, if $\alpha < \gamma$ then there is $n \in \mathbb{N}$ with $a(n) \geq \alpha$ since $\gamma = \sup_n(a(n)+1)$, and $p_n \in p_n b_n \subseteq Y^\alpha$ for such n, hence $Q^\alpha(Y, 0) \neq \{0\}$. This proves $\operatorname{nc}(Y, 0) \geq \gamma$.

We now verify (A) and (B) with induction on $\alpha \leq \gamma$. Clearly (A) and (B) hold for $\alpha = 0$. Now let $\beta \in (0, \gamma]$, and suppose (A) and (B) to be known for $\alpha < \beta$.

Case 1: *β is a successor ordinal.*

Let $\beta = \alpha + 1$, then we have to prove that

$$Q(Y^\alpha, 0) = Y^{\alpha+1} \quad \text{and} \quad Q(Y^\alpha \cup \{b\}, 0) = Y^{\alpha+1} \cup \{b\}.$$

Since $b \notin Y^\alpha$ our task simplifies to proving that

$$Y^{\alpha+1} \subseteq Q(Y^\alpha, 0), \quad \text{and} \quad b \in Q(Y^\alpha \cup \{b\}, 0), \quad \text{and}$$

$$Q(Y^\alpha \cup \{b\}, 0) \subseteq Y^{\alpha+1} \cup \{b\}.$$

Our assumption on the Y_n implies that

$$Q(Y^\alpha \cap X_n, 0) = \begin{cases} Q(Q^\alpha(Y_n, 0) \cup p_n b_n, 0) = Q^{\alpha+1}(Y_n, 0) \cup p_n b_n, \\ \quad \text{if } a(n) < \alpha, \\ Q(\{a\} \cup p_n b_n, 0) = \{0\}, \\ \quad \text{if } a(n) = \alpha, \\ \{0\}, \\ \quad \text{if } a(n) > \alpha. \end{cases}$$

It immediately follows that $Y^{\alpha+1} \subseteq Q(Y^\alpha, 0)$. Also, for each $n \in \mathbb{N}$ the following holds: $X_n - \{0\}$ is clopen in $X - \{0\}$ by (II), hence if K is a clopen subset of $Y^\alpha \cap T_n$ that does not contain 0 then K is clopen in Y^α. It follows that $Q(Y^\alpha \cup \{b\}, 0) \subseteq Y^{\alpha+1} \cup \{b\}$. It remains to show that $b \in Q(Y^\alpha \cup \{b\}, 0)$: Let U be a clopen subset of $Y^\alpha \cup \{b\}$ that contains 0. Since $\lim_n p_n = 0$ the fact that $a : \mathbb{N} \to [0, \gamma)$ is nondecreasing with $\alpha < \gamma = \sup_n(a(n)+1)$ implies that there is $m \in \mathbb{N}$ with $p_n \in U$ and $a(n) \geq \alpha$ for all $n \geq m$. Now $p_n b_n \subseteq Y^\alpha$ for all n with $a(n) \geq \alpha$, hence $p_n b_n \subseteq U$ for all $n \geq m$. Since $\lim_n b_n = b$ it follows that $b \in U$.

Case 2: *β is a successor cardinal.*

We have to prove that $Y^\beta = \bigcap_{\alpha<\beta} Y^\alpha$: Just observe that $Y = Y^0 \subseteq \bigcup_n X_n$, and check that $X_n \cap Y^\beta = X_n \cap \bigcup_{\alpha<\beta} Y^\alpha$ separately for n with $a(n) \geq \beta$ and for n with $a(n) < \beta$; this is easy.

A problem

Lelek has asked if (**) of the Introduction holds if X is a metrizable continuum that is Suslinean, i.e., that does not have an uncountable pairwise disjoint collection of nondegenerate subcontinua, [2, p. 8]. Since rational continua are Suslinean our example also answers this question in the negative. We therefore ask the weaker question of whether (*) of the Introduction holds if X is a metrizable Suslinean continuum.

References

[1] A. Lelek, On the topology of curves I, Fund. Math. 67 (1970) 359-367.
[2] A. Lelek, Some problems concerning curves, Colloq. Math. 23 (1971) 93-98.

Notes by the editor

This paper was originally submitted to Houston Journal of Mathematics on June 2, 1981. It was refereed and accepted for publication but van Douwen never submitted a revised version of his manuscript. The editor of Houston Journal of Mathematics has authorized me to publish the paper here and he has provided me with a copy of the report of the referee. I have made those changes and improvements suggested by the referee that I thought were appropriate.